数理解析研究所講究録 1232 巻 2001 年 49-77

Obstruction to and Deformation of Lagrangian Intersection Floer Cohomology

Hiroshi Ohta*
Graduate School of Mathematics, Nagoya University,
Chikusa, Nagoya, 464-8602, Japan,
E-mail: ohta@math.nagoya-u.ac.jp

This is a short survey about obstruction theory to and deformation theory of Lagrangian intersection Floer cohomology, developed in our joint paper [FOOO]. The obstruction to define Lagrangian intersection Floer cohomology is systematically investigated and a system of the obstruction classes are constructed. Also, a filtered A_∞ algebra associated to Lagrangian submanifold is constructed and by using it, deformation of the Lagrangian submanifold and Floer cohomology is described in terms of the notion of filtered A_∞ algebra. Moreover, some applications of our theory to concrete problems in symplectic geometry are discussed.

CONTENTS

---

*Partially supported by Grant in Aid for Scientific Research no 12640066, the Ministry of Education, Science, Sports and Culture, Japan.

§0. Introduction

In symplectic geometry, there are two kinds of Floer cohomologies. One is the absolute version and the other is the relative version. The absolute version is related to the periodic Hamiltonian systems and the Arnold conjecture for the fixed point sets of the Hamiltonian diffeomorphisms of symplectic manifold. The relative version is related to Lagrangian intersection theory. Our Floer cohomology we will discuss here is the relative one. Roughly speaking, from the point of view of Morse theory, the generators of the Morse cochain complex in the absolute case are the set of fixed points of a Hamiltonian diffeomorphism. The spaces of the gradient trajectories, which are needed to define the coboundary operators, are moduli spaces of J-holomorphic maps from infinite cylinder such that two end points converge to corresponding two fixed points. By the removable singularity theorem for J-holomorphic maps, the space can be regarded as moduli space of J-holomorphic 2-spheres. The fundamental theory of moduli space of J-holomorphic curves *without boundary* is now established and we can define the Floer cohomology in absolute case for general symplectic manifolds. See [FO], [LT], [B] etc. However, in the relative case, we have new problems and difficulties which do not appear in the absolute case. The generators of the cochain complex correspond to intersection points of two Lagrangian submanifolds. To define coboundary operators, we have to study moduli spaces of J-holomorphic maps from infinite strip or disc with Lagrangian boundary condition. In particular, we have to study moduli space of J-holomorphic curves *with boundary*. If we define, as in a usual way, the "coboundary operator" δ in Lagrangian intersection Floer theory by counting the number of certain components of moduli spaces of J-holomorphic discs, δ does *not* satisfy $\delta \circ \delta = 0$ in general. This is essentially because the phenomena that holomorphic disc bubbles off at a point of the boundary of holomorphic disc happens. This is real codimension one phenomena. (See §1). This is the main trouble to overcome. We will study the obstruction to $\delta \circ \delta = 0$ systematically. Moreover, in the case of our obstructions vanish, we will develop a deformation theory of Lagrangian intersection Floer cohomologies. The obstruction and the deformation are described in terms of certain homological algebra, so called A_∞-algebra. Strictly speaking, we introduce and use a notion of *filtered* A_∞-algebra.

§1. Preliminaries and problems to overcome.

Let (M, ω) be a smooth symplectic manifold with real dimension $2n$ and L_0, L_1 closed Lagrangian submanifolds of M. We assume that our Lagrangian submanifold is always orientable. Although it is enough that we assume that L_0 and L_1 intersect cleanly in Bott's sense, we assume here that L_0 and L_1

intersect transversally for simplicity.

First of all, we briefly explain our setting. Consider the path space

$$\Omega(L_0, L_1) = \{\ell : [0,1] \to M \mid \ell(0) \in L_0, \ell(1) \in L_1\}.$$

We choose and fix a base point $\ell_0 \in \Omega(L_0, L_1)$ on each connected component of $\Omega(L_0, L_1)$. We now describe a covering space of the component $\Omega_{\ell_0}(L_0, L_1)$ of $\Omega(L_0, L_1)$ that contains ℓ_0. Consider the set of all pairs (ℓ, w) satisfying:

(1.1.1) $w(0, \cdot) = \ell_0$,
(1.1.2) $w(\tau, 0) \in L_0$, $w(\tau, 1) \in L_1$ for all $0 \le \tau \le 1$,
(1.1.3) $w(1, \cdot) = \ell$,

where $w : [0,1] \times [0,1] \to M$. We define an equivalence relation on this set as follows: First, we consider any closed loop

$$c : S^1 \to \Omega_{\ell_0}(L_0, L_1)$$

which will also define a pair of closed loops in L_0 and L_1 for $t = 0, 1$ respectively. Noting that every symplectic vector bundle over S^1 is trivial, the bundle c^*TM over $S^1 \times [0,1]$ is symplectically trivial. Therefore any such trivialization defines two closed loops of Lagrangian subspaces

$$\alpha_0, \alpha_1 : S^1 \to \Lambda(\mathbf{C}^n)$$

by

$$\alpha_0(\tau) = T_{c(\tau,0)}L_0, \qquad \alpha_1(\tau) = T_{c(\tau,1)}L_1,$$

in the trivialization. Here $\Lambda(\mathbf{C}^n)$ denotes the space of all Lagrangian subspaces in $\mathbf{C}^n$. We fix any such trivialization

$$\Psi : c^*TM \to S^1 \times [0,1] \times \mathbf{C}^n$$

and denote by $\mu_\Psi(\alpha_i)$ the Maslov index of the loop α_i in the trivialization Ψ. One can find that the difference

$$\mu_\Psi(\alpha_1) - \mu_\Psi(\alpha_0)$$

is independent of the choice of trivialization Ψ but depends only on the loop c. We denote this common number by $\mu(c)$ and call it the *Maslov index* of the loop c in $\Omega(L_0, L_1)$. It defines an integer valued homomorphism

$$\mu : \pi_1(\Omega_{\ell_0}(L_0, L_1), \ell_0) \to \mathbf{Z}. \tag{1.2}$$

Using (1.2) and the symplectic form ω, we define an equivalence relation $\sim$ on the set of all pairs (ℓ, w) satisfying (1.1). We denote by $w\#w'$ the concatenation of w and w' along ℓ, which will define a loop in $\Omega_{\ell_0}(L_0, L_1)$ based at ℓ_0.

Definition 1.3. We say that (ℓ, w) is equivalent to (ℓ, w') and write $(\ell, w) \sim (\ell, w')$ if the following conditions are satisfied

$$\int_w \omega = \int_{w'} \omega \quad \text{i.e.} \quad \int_{\overline{w}\#w'} \omega = 0 \tag{1.3.1}$$

$$\mu(\overline{w}\#w') = 0 \tag{1.3.2}$$

where $\overline{w}$ is the disc w with the opposite orientation.

We define a covering space of $\Omega_{\ell_0}(L_0, L_1)$ by

$$\widetilde{\Omega}_{\ell_0}(L_0, L_1) = \{(\ell, w) \mid \text{ satisfying (1.1)}\}/\sim .$$

We denote by $[\ell, w]$ the equivalence class of (ℓ, w). Now we define a functional $\mathcal{A} : \widetilde{\Omega}_{\ell_0}(L_0, L_1) \to \mathbf{R}$ by

$$\mathcal{A}([\ell, w]) = \int w^*\omega. \tag{1.4}$$

A simple standard calculation shows that the set of critical points of $\mathcal{A}$ on $\widetilde{\Omega}_{\ell_0}(L_0, L_1)$ are those $[\ell_p, w]$ where $\ell_p : [0,1] \to M$ is the constant path corresponding to an intersection point $p \in L_0 \cap L_1$. We denote by $Cr_{\ell_0}(L_0, L_1)$ the set of all critical points of

$$\mathcal{A} : \widetilde{\Omega}_{\ell_0}(L_0, L_1) \to \mathbf{R},$$

and put $Cr(L_0, L_1) = \cup_{\ell_0} Cr_{\ell_0}(L_0, L_1)$. We next study the gradient lines of $\mathcal{A}$. As usual, we fix a compatible almost complex structure J on M and consider the induced Riemannian metric $g_J := \omega(\cdot, J\cdot)$. This will in turn induce an L^2-metric on $\widetilde{\Omega}_{\ell_0}(L_0, L_1)$. We now define the moduli space $\widetilde{\mathcal{M}}_J([\ell_p, w], [\ell_q, w'])$ as follows: $\widetilde{\mathcal{M}}_J([\ell_p, w], [\ell_q, w'])$ is the set of maps

$$u : \mathbf{R} \times [0,1] \to M$$

with

$$u(\mathbf{R} \times \{0\}) \subset L_0, \quad u(\mathbf{R} \times \{1\}) \subset L_1, \tag{1.5.1}$$

(1.5.2) u satisfies

$$\begin{cases} \dfrac{\partial u}{\partial \tau} + J\dfrac{\partial u}{\partial t} = 0 \\ \lim_{\tau \to -\infty} u(\tau, t) = p, \qquad \lim_{\tau \to +\infty} u(\tau, t) = q \end{cases}$$

(1.5.3) $w \# u \sim w'$.

Here $w \# u$ is the obvious concatenation of w and u along the constant path ℓ_p.

From now on, we will suppress J from various notations whenever possible. Then we have the following:

Proposition 1.6. *There exists a map $\mu : Cr(L_0, L_1) \to \mathbf{Z}$ such that the space $\widetilde{\mathcal{M}}([\ell_p, w], [\ell_q, w'])$ has a Kuranish structure of dimension $\mu([\ell_q, w']) - \mu([\ell_p, w])$. We also assume that the pair (L_0, L_1) is relatively spin. Then the space will carry an orientation in the sense of Kuranish structure.*

Remark 1.7. (1) The space $\widetilde{\mathcal{M}}([\ell_p, w], [\ell_q, w'])$ is not a smooth manifold, in general. This trouble comes from the transversality problem. In order to overcome this problem, we have now an established machinery, so called *Kuranish structure* introduced in [FO]. We do not explain the notion of Kuranishi structure here. See [FO] and [FOOO]. When we use Kuranishi structure, the "(virtual) fundamental class" is defined only over $\mathbf{Q}$, not $\mathbf{Z}$. So we can not work over $\mathbf{Z}/2\mathbf{Z}$ coefficient in general. In this sense, we can not avoid the orientation problem. In this note, we do not mention about the transversality problem no more.

(2) The definition of relatively spin will be given in §2. We should note that this space is not always orientable, in general. In the absolute version of Floer cohomology for Hamiltonian diffeomorphism, the corresponding spaces of gradient trajectories (or connecting orbits) which are used to define the coboundary operator are always orientable and have a canonical orientation induced by an almost complex structure. The reason why we can not expect the space of connecting orbits for the Lagrangian intersection Floer cohomology is basically that the almost complex structure does not preserve the Lagrangian boundary condition.

We have an $\mathbf{R}$-action on $\widetilde{\mathcal{M}}([\ell_p, w], [\ell_q, w'])$ defined by the translation along the τ-direction, and put

$$\mathcal{M}([\ell_p, w], [\ell_q, w']) = \widetilde{\mathcal{M}}([\ell_p, w'], [\ell_p, w])/\mathbf{R}.$$

The standard Floer's cochain "complex" $(CF(L_0, L_1), \delta_0)$ (actually this is not a complex, in general) is defined as

Definition 1.8. We assume that the pair (L_0, L_1) is relatively spin.

$$CF^k(L_0, L_1) = \widehat{\bigoplus_{\substack{[\ell_p, w] \in Cr(L_0, L_1), \\ \mu([\ell_p, w]) = k}}} \mathbf{Q}[\ell_p, w] \tag{1.8.1}$$

$$\delta_0[\ell_p, w] = \sum_{\mu[\ell_q, w] = \mu[\ell_p, w]+1} \#(\mathcal{M}([\ell_p, w], [\ell_q, w']))[\ell_q, w']. \tag{1.8.2}$$

Here $\widehat{\oplus}$ means an appropriate completion. Since we use the Kuranishi structure of $\mathcal{M}([\ell_p, w], [\ell_q, w'])$, the number in the right hand side of (1.8.2) is a rational number.

For the absolute case of Floer cohomology, similar constructions have been used. However, there is a crucial difference for the case of Lagrangian intersections from the absolute case: The boundary $\partial\mathcal{M}([\ell_p, w], [\ell_q, w']))$ consists of more than

$$\bigcup_{\mu([\ell_r, w'']) = \mu([\ell_p, w])+1} \mathcal{M}([\ell_p, w], [\ell_r, w'']) \times \mathcal{M}([\ell_r, w''], [\ell_q, w']).$$

More precisely, the compactification of $\mathcal{M}([\ell_p, w], [\ell_q, w'])$ has extra codimension one components other than those of "split connecting orbits". The extra components come from bubbling-off discs. From the index formula, we know that bubbling-off spheres are phenomena of real codimension at least two (complex codimension at least one), while bubbling-off discs is of real codimension one in general. Therefore $\delta_0 \circ \delta_0 \neq 0$, in general. Thus we have an obstruction to define Floer cohomology in the relative case.

Thus we can summarize our problems (modulo transversality problems) to overcome as follows.

• **Obstruction problem:** We have to study the obstruction to $\delta_0 \circ \delta_0 = 0$ systematically.

• **Orientation problem:** Find a condition for the moduli space of J holomorphic curves with boundary (Lagrangian boundary condition) to be orientable. Moreover we have to discuss the problems about the orientations on various moduli spaces carefully.

§2. Orientation and obstruction classes.

To state our results, let us introduce some notations. We have two important group homomorphisms from $\pi_2(M, L)$:

$$\mathcal{A} : \pi_2(M, L) \to \mathbf{R}, \quad \text{and} \quad \mu_L : \pi_2(M, L) \to \mathbf{Z}. \tag{2.0}$$

Here $\mathcal{A}$ is defined by $\mathcal{A}(\beta) = \omega(\beta)$ for $\beta \in \pi_2(M, L)$, which is called symplectic area (or energy) and μ_L is called the Maslov index. We can define μ_L in a way similar to μ in §1. We omit the precise definition of μ_L. See [Oh], for example. We note that μ_L is always even when L is orientable.

Definition 2.1. For $\beta \in \pi_2(M, L)$, we denote by $\mathcal{M}_{k+1}(L, \beta)$ the set of all isomorphism classes of genus zero stable J holomorphic maps $w : D^2 \to M$ with $k+1$ marked points on the boundary ∂D^2 such that $w(\partial D^2) \subset L$ and $[w] = \beta$. We denote by $\mathcal{M}^{\mathrm{main}}_{k+1}(L, \beta)$ the component which corresponds to that the ordering of the marked points are cyclic. We call it a *main component.*

Remark 2.2. As usual, the stability means that the automorphism group of $((D; z_0, \dots, z_{k+1}), w)$ is finite. Here the automorphism $\varphi : D \to D$ is the biholomorphic map such that $w \circ \varphi = w$ and $\varphi(z_i) = z_i$. Strictly speaking, we need impose extra interior marked points, (for example to handle sphere bubbles), but we omit these points here.

Then we have

Proposition 2.3. *$\mathcal{M}_{k+1}(L, \beta)$ has a Kuranishi structure of real dimension $n + \mu_L(\beta) - 3 + k + 1$. Here $n = \dim L$ and 3 is dimension of $\mathrm{Aut}(D^2) \cong PSL_2(\mathbf{R})$.*

As for the orientability of this space, we have to introduce a notion of relative spin Lagrangian submanifold.

Definition 2.4. (1) An orientable Lagrangian submanifold L in (M, ω) is called *relative spin* if there exists a class $st \in H^2(M; \mathbf{Z}/2\mathbf{Z})$ such that $w_2(TL) \equiv st|_L$ in $H^2(L; \mathbf{Z}/2\mathbf{Z})$. In particular, when L is spin, it is relative spin.

(2) A pair of Lagrangian submanifolds (L_0, L_1) is *relative spin* if there exits a class $st \in H^2(M; \mathbf{Z}/2\mathbf{Z})$ such that $w_2(TL_0) \equiv st|_{L_0}$ and $w_2(TL_1) \equiv st|_{L_1}$ simultaneously.

Now let us state our results. Firstly, we have the following theorem about the orientation problem.

Theorem 2.5. *We denote by $\widetilde{\mathcal{M}}$ the space of all J holomorphic maps $w : D^2 \to M$ with $w(\partial D) \subset L$. Assume that L is relative spin. Then $\widetilde{\mathcal{M}}$ is orientable. The orientation is given by the choice of an orientation of L, $st \in H^2(M; \mathbf{Z}/2\mathbf{Z})$ and a spin structure on $TL \oplus V|_{L^{(2)}}$, where V is the vector bundle on 3-skeleton of M determined by st and $L^{(2)}$ stands for 2-skeleton of L. Moreover, if a pair of Lagrangian submanifold (L_0, L_1) is relative spin, then $\widetilde{\mathcal{M}}([\ell_p, w], [\ell_q, w'])$ is orientable. The orientation is given by the choice of orientations of L_0 and L_1, $st \in H^2(M; \mathbf{Z}/2\mathbf{Z})$ and spin structures on $TL_0 \oplus V|_{L_0^{(2)}}$ and $TL_1 \oplus V|_{L_1^{(2)}}$*

This theorem can be proved by some gluing argument on the indices of families of linearized Dolbeault operators and an elementary topological argument.

As for the obstruction problem, we can show the following theorem.

Theorem 2.6. *Let L be an oriented Lagrangian submanifold of (M,ω). Assume that L is relative spin. Then we have the series of homology classes $\{o_k(L)\}_{k=1,2,\ldots}$ of L which satisfy the following significances:*

(1) $o_k(L) \in H_{n+\mu_L(\beta_k)-2}(L;\mathbf{Q})$. More precisely, $o_k(L)$ is in

$$\mathrm{Ker}\ (H_{n+\mu_L(\beta_k)-2}(L;\mathbf{Q}) \longrightarrow H_{n+\mu_L(\beta_k)-2}(M;\mathbf{Q})).$$

(2) $o_k(L)$ is defined if $o_j(L) = 0$ in $H_*(L;\mathbf{Q})$ for every $j < k$.

(3) If all $o_k(L)$ vanish, then we can define the Floer cohomology $HF(L,L)$ by deforming the coboundary operators.

(4) Assume that a pair of Lagrangian submanifolds (L_0, L_1) is relative spin. (Then we can define the series $\{o_k(L_0)\}$ and $\{o_k(L_1)\}$.) If all $o_k(L_0)$ and $o_k(L_1)$ vanish, then we can define the Floer cohomology $HF(L_0,L_1)$ by deforming coboundary operators.

We call $o_k(L)$ an *obstruction class.*

Remark 2.7. (1) Let us explain what the β_k's are. These are elements of $\pi_2(M,L)$ such that β_k is represented by a J-holomorphic disc. Then Gromov's compactness theorem implies that for each $C \geq 0$ the number of the set

$$\{\beta \in \pi_2(M,L) \mid \beta \text{ is represented by a } J \text{ holomorphic disc and } \mathcal{A}(\beta) \leq C\}$$

is finite. Therefore we have a partial order on the set of all $\beta \in \pi_2(M,L)$ which are represented by J holomorphic discs by the energy $\mathcal{A}$. Namely, we have

$$0 = \mathcal{A}(\beta_0) < \mathcal{A}(\beta_1) \leq \mathcal{A}(\beta_2) \leq \cdots \leq \mathcal{A}(\beta_{k-1}) \leq \mathcal{A}(\beta_k) \leq \cdots.$$

Here $\beta_0 = 0$ corresponds to constant maps.

(2) Taking (1) in Theorem 2.6 into account, we find that if $\mu_L(\beta_k) \geq 3$ for all k, then the obstruction classes automatically vanish. This condition was essentially used in the earlier work by Y-G Oh [Oh]. Here he defined Floer cohomology over $\mathbf{Z}/2\mathbf{Z}$ under some additional assumption which guarantees the trouble about the transversality problem does not happen, so the Kuranishi structure is not necessary into account in his case.

(3) We do not specify the coefficient ring of Floer cohomology here. See §4.

§3. Construction of the obstruction classes.

3.A) On orientations.

Before we explain the idea of construction of the obstruction classes, we like to mention a little bit about the orientations on various moduli spaces which will be used. From now on, we always assume that L is relative spin. We have an evaluation map at the marked point z_j

$$ev_j : \mathcal{M}_{k+1}(L, \beta) \longrightarrow L$$

for each $j = 0, 1, \dots, k$, defined by $ev_j((w; z_0, \dots, z_k)) = w(z_j)$. Let P_j be an oriented chain in L. We put $\deg P_j = n - \dim P_j$. We take a fibre product (in the sense of Kuranishi structure)

$$\mathcal{M}_{k+1}(L, \beta)_{(ev_1, \dots, ev_k)} \times (P_1 \times \cdots \times P_k).$$

Then, by using the orientations on $\mathcal{M}_{k+1}(L, \beta)$ (defined by Theorem 2.5), P_j's and L, we can define the fibre product orientation on it. But we use a different orientation from the fibre product orientation.

Definition 3.1. We put

$$\mathcal{M}_1(\beta; P_1, \dots, P_k) = (-1)^{\epsilon} \mathcal{M}_{k+1}(L, \beta)_{(ev_1, \dots, ev_k)} \times (P_1 \times \cdots \times P_k)$$

Here ϵ is given by

$$\epsilon = (n+1) \sum_{j=1}^{k-1} \sum_{i=1}^{j} \deg P_i.$$

If we take the fibre product iteratively, we can rewrite the right hand side as

$$\begin{aligned} \mathcal{M}_1(\beta; P_1, \dots, P_\ell) = & (-1)^{\sum_{k=1}^{\ell-1} \sum_{j=1}^{k} \deg P_j} \\ & \Big(\cdots ((\mathcal{M}_{\ell+1}(\ \beta\ )_{ev_1} \times_{f_1} P_1)_{ev_2} \times_{f_2} P_2) \times \cdots \Big)_{ev_\ell} \times_{f_\ell} P_\ell. \end{aligned}$$

The "feeling" of the sign is an effect from the marked points. Roughly speaking, there might be two conventions when we consider the effect of the marked points. One is that we put all the parameters which describe the marked points on a "one side" in the fibre product. But we use another convention. We put the one dimensional parameters which describe the each marked point "one by one" in the fibre product. We call our convention *BARAMAKI* way. (We call the first one *HAKIYOSE* way.) If we change the ordering of the marked points, then we have another connected component (which are homeomorphic to each other). But the orientation might be changed. Under our orientation in Definition 3.1, we can find that the change is given by the following.

Proposition 3.2. *Let σ be the transposition element $(i, i+1)$ in the k-th symmetric group S_k. $(i = 1, \dots, k-1)$. Then the action of σ on*

$$\mathcal{M}_1(\beta; P_1, \dots, P_i, P_{i+1}, \dots, P_k)$$

by changing the order of marked points is described by following.

$$\begin{aligned}&\sigma(\mathcal{M}_1(\beta; P_1, \dots, P_i, P_{i+1}, \dots, P_k)) \\ &= (-1)^{(\deg P_i + 1)(\deg P_{i+1}+1)} \mathcal{M}_1(\beta; P_1, \dots, P_{i+1}, P_i, \dots, P_k).\end{aligned}$$

Now we explain the idea of the construction of our obstruction classes. We construct $o_k(L)$ inductively.

3.B) The first obstruction class.

We consider the space $\mathcal{M}_1(\beta_1)$ with the evaluation map ev_0,

$$ev_0 : \mathcal{M}_1(\beta_1) \longrightarrow L.$$

Note that $\mathcal{A}(\beta_1)$ is the minimal (non zero) area. Hence for an element in $\mathcal{M}_1(\beta_1)$, the bubbling off phenomena does not happen. Therefore $\partial\mathcal{M}_1(\beta_1) = \emptyset$. Thus $ev_0(\mathcal{M}_1(\beta_1))$ is a cycle in L, so defines a homology class. We define the first obstruction class $o_1(L)$ by the homology class:

$$o_1(L) = [ev_0(\mathcal{M}_1(\beta_1))].$$

The degree is given by $n + \mu_L(\beta_1) - 2$ because of Proposition 2.3.

3.C) The higher obstruction classes.

We suppose that $o_j(L) = 0$ for all $1 \le j \le k-1$. Under this situation, we are going to construct the k-th obstruction class $o_k(L)$. By the assumption we have bounding chains $\mathcal{B}_j = \mathcal{B}_j(L) \subset L$ such that

$$\partial\mathcal{B}_j(L) = (-1)^{n+1} o_j(L).$$

Here the orientation on $o_j(L)$ is given by the orientation on $\mathcal{M}_1(\beta_*; \mathcal{B}_*, \dots, \mathcal{B}_*)$ defined in Definition 3.1 and the orientation on $\mathcal{B}_j(L)$ is given by changing the boundary orientation by $(-1)^{n+1}$. This sign plays an important role in the later argument. We put

$$\begin{aligned}&\mathcal{M}_1(\beta_k; \mathcal{B}_{i_1}, \dots, \mathcal{B}_{i_m}) \\ &= (-1)^{\epsilon_1} \mathcal{M}_{m+1}(L; \beta_k - \beta_{i_1} - \dots - \beta_{i_m})_{(ev_{i_1}, \dots, ev_{i_m})} \times (\prod_{\ell=1}^{m} \mathcal{B}_{i_\ell})\end{aligned}$$

for $i_1, \dots, i_m < k$. Here ϵ_1 is given by the rule in Definition 3.1, that is

$$\epsilon_1 = (n+1)\frac{m(m-1)}{2},$$

because $\deg \mathcal{B}_{i_\ell} \equiv \mu_L(\beta_{i_\ell}) + 1 \equiv 1 \pmod 2$. (Note that since L is orientable, μ_L is always even.) It is easy to see that the dimension of $\mathcal{M}_1(\beta_k; \mathcal{B}_{i_1}, \dots, \mathcal{B}_{i_m})$ is given by $n + \mu_L(\beta_k) - 2$. (Recall μ_L is a group homomorphism.) Then we define

Definition 3.3.

$$o_k(L) = \sum_{\substack{m=0,1,2,\dots \\ i_1,\dots,i_m<k \\ \beta_k - \sum_{j=1}^m \beta_{i_j} \in G_+(L)}} \frac{1}{m!}(ev_0(\mathcal{M}_1(\beta_k; \mathcal{B}_{i_1}, \dots, \mathcal{B}_{i_m})),$$

where $G_+(L)$ stands for the subset of $\pi_2(M, L)$ whose elements are represented by J holomorphic discs. Note that the right hand side is a finite sum.

Then we have a chain in L defined by $o_k(L)$. What we have to show is that $o_k(L)$ defines a *cycle* . We can show the following.

Proposition 3.4. $\partial o_k(L) = \emptyset$.

If we ignore the sign problems, the proof is, in a sense, easy. That is, we have two kinds of boundaries of $\mathcal{M}_1(\beta_k; \mathcal{B}_{i_1}, \dots, \mathcal{B}_{i_m})$ like as

$$\begin{aligned}
&\partial \mathcal{M}_1(\beta_k; \mathcal{B}_{i_1}, \dots, \mathcal{B}_{i_m}) \\
&= (-1)^{\epsilon_1}\Big(\partial \mathcal{M}_{m+1}(L; \beta_k - \beta_{i_1} - \cdots - \beta_{i_m})_{(ev_{i_1},\dots,ev_{i_m})} \times (\prod_{\ell=1}^m \mathcal{B}_{i_\ell}) \\
&\quad \coprod \prod_{\ell=1}^m (-1)^{n+m+nm} \mathcal{M}_{m+1}(L; \beta_k - \beta_{i_1} - \cdots - \beta_{i_m})_{(ev_{i_1},\dots,ev_{i_m})} \times \\
&\qquad\qquad (\mathcal{B}_{i_1} \times \cdots \times \partial \mathcal{B}_{i_\ell} \times \cdots \times \mathcal{B}_{i_m})\Big).
\end{aligned}$$

The first type boundaries correspond to the bubbling off J holomorphic dics $\partial \mathcal{M}_{m+1}(L; \beta_k - \beta_{i_1} - \cdots - \beta_{i_m})$ and the second type boundaries correspond to the case when the bounding chain $\mathcal{B}_{i_\ell}$ goes to $\partial \mathcal{B}_{i_\ell} = (-1)^{n+1} o_{i_\ell}(L)$. We take the summation in Definition 3.3 over all "lower" strata of moduli spaces. Therefore these two kinds of boundaries cancel each other. The non trivial issue is that they cancel each other with *sign.* That is, we have to show that the orientations on these two kinds of boundaries are opposite. But in this note, we omit the proof.

Now let (L_0, L_1) be a pair of relative spin Lagrangian submanifolds. If all obstructions $o_k(L_0)$ and $o_k(L_1)$ vanish, then, as we state in Theorem 2.6.(4), we can define Floer cohomology $HF(L_0, L_1)$ by deforming the coboundary operators as follows. By assumption, we have bounding chains $\mathcal{B}_{0,*} = \mathcal{B}_*(L_0) \subset L_0$ and $\mathcal{B}_{1,*} = \mathcal{B}_*(L_1) \subset L_1$. By imposing marked points on the boundaries of the strip $[0,1] \times \mathbf{R}$ in (1.5) (ℓ marked points on $\{0\} \times \mathbf{R}$ and m marked points on $\{1\} \times \mathbf{R}$), we can define the fibre product like as

$$\begin{aligned}&\mathcal{M}([\ell_p, w], [\ell_q, w']; \mathcal{B}_{0,i_1}, \dots, \mathcal{B}_{0,i_\ell}; \mathcal{B}_{1,j_1}, \dots, \mathcal{B}_{1.j_m})\\ &= (-1)^{\epsilon_2} \mathcal{M}_{\ell,m}([\ell_p, w], [\ell_q, w'])_{(ev_1^0,\dots,ev_\ell^0,ev_1^1,\dots,ev_m^1)} \times (\prod_{k=1}^{\ell} \mathcal{B}_{0,i_k} \times \prod_{k=1}^{m} \mathcal{B}_{1,j_k}),\end{aligned}$$

where ϵ_1 is given by

$$\epsilon_1 = (n+1)(\sum_{k=1}^{\ell+m-1} \sum_{j=1}^{k} 1) = \frac{(n+1)(\ell+m-1)(\ell+m)}{2},$$

which is consistent with the rule in Definition 3.1. Now we define an operator $\delta_{\beta_{0,i_1},\dots,\beta_{0,i_\ell};\beta_{1,j_1},\dots,\beta_{1,j_m}}$ by

$$\begin{aligned}&\langle \delta_{\beta_{0,i_1},\dots,\beta_{0,i_\ell};\beta_{1,j_1},\dots,\beta_{1,j_m}} [\ell_p, w], [\ell_q, w'] \rangle\\ &:= \# \mathcal{M}([\ell_p, w], [\ell_q, w']; \mathcal{B}_{0,i_1}, \dots, \mathcal{B}_{0,i_\ell}; \mathcal{B}_{1,j_1}, \dots, \mathcal{B}_{1,j_m}).\end{aligned}$$

Then we can show

Proposition 3.5. *We define our modified coboundary operator δ by*

$$\delta := \sum_{\ell,m} \sum_{\beta_{0,i_1},\dots,\beta_{0,i_\ell};\beta_{1,j_1},\dots,\beta_{1,j_m}} \frac{1}{\ell! m!} \delta_{\beta_{0,i_1},\dots,\beta_{0,i_\ell};\beta_{1,j_1},\dots,\beta_{1,j_m}}.$$

Then it satisfies $\delta \circ \delta = 0$.

§4. A_∞-deformation of Lagrangian submanifold.

We should note that the constructions in the previous section depend, *a priori*, on various choices of the bounding chains, the almost complex structures, and the Kuranishi structures. In this note, we only discuss dependence on the bounding chains. (As for our conclusions about "independence", see Theorem 5.19 and Theorem 5.20.) To do this we use language of certain homological algebras. The key point is that we have to work at *chain level*, not homological level. Firstly, we construct a *filtered A_∞ algebra* associated to a relative spin Lagrangian submanifold L. Here we note that we do *not* assume the obstruction classes of L vanish.

4.A) A filtered A_∞ algebra.

First of all, we introduce the notion of *filtered A_∞ algebra* [FOOO]. Let us introduce the universal Novikov ring.

Definition 4.1. ([FOOO]). Let T and e be two formal variables. *The universal Novikov ring* Λ_{nov} is the totality of all formal sums $\sum a_i T^{\lambda_i} e^{n_i}$ such that

(4.1.1) $a_i \in \mathbf{Q}$, $\lambda_i \in \mathbf{R}$ and $n_i \in \mathbf{Z}$.

(4.1.2) $\lim_{i\to\infty} \lambda_i = \infty$.

We define its subset $\Lambda_{0,nov}$ by

$$\Lambda_{0,nov} = \left\{ \sum a_i T^{\lambda_i} e^{n_i} \mid \lambda_i \geq 0, \text{ and } n_i = 0 \text{ if } \lambda_i = 0 \right\}.$$

We define the product of elements of Λ_{nov} in an obvious way. Then Λ_{nov} is a commutative ring with the unit 1, and $\Lambda_{0,nov}$ is its subring. We define the grading by

$$\deg T^\lambda e^n = 2n.$$

Roughly speaking, λ_i stands for a filtration and n_i for a grading. Geometrically, λ_i corresponds to an energy $\mathcal{A}$ in §1 or §2, and n_i corresponds to the Maslov index. When we consider a pair of Lagrangian submanifolds, we use Λ_{nov} as well. (It might be helpful to keep the geometric back ground in your mind.) We remark the $\Lambda_{0,nov}$ is a local ring with the maximal ideal

$$\Lambda_{0,nov}^+ = \{ \sum a_i T^{\lambda_i} e^{n_i} \in \Lambda_{nov} \mid \lambda_i > 0 \}$$

such that $\Lambda_{0,nov}/\Lambda_{0,nov}^+ \cong \mathbf{Q}$. So when we reduce the coefficient ring to $\Lambda_{0,nov}/\Lambda_{0,nov}^+ \cong \mathbf{Q}$, then we do not have filtrations. See (4.0) below.

Let $\oplus_{m\in\mathbf{Z}} C^m$ be a free graded $\Lambda_{0,nov}$ module. There is a filtration $F^\lambda C^m$ on C^m ($\lambda \in \mathbf{R}_{\geq 0}$), such that

(4.2.1) $F^\lambda C^m \subset F^{\lambda'} C^m$ if $\lambda > \lambda'$.

(4.2.2) $T^{\lambda_0} \cdot F^\lambda C^m \subset F^{\lambda+\lambda_0} C^m$.

(4.2.3) $e^k C^m \subset C^{m+2k}$.

(4.2.4) C^m is complete with respect to the filtration.

(4.2.5) C^m has a basis $\mathbf{e}_i$ such that $\mathbf{e}_i \in F^0 C^m$ and $\mathbf{e}_i \notin F^\lambda C^m$ for $\lambda > 0$.

We denote by C the completion of $\oplus_{m\in\mathbf{Z}} C^m$ with respect to the filtration. (4.2.3) means that the degree of e is 2. We put $(C[1])^m = C^{m+1}$ and

$$B_k(C[1]) = \bigoplus_{m_1,\cdots,m_k} (C[1])^{m_1} \otimes \cdots \otimes (C[1])^{m_k}.$$

Suppose that we have a sequence of maps $\mathbf{m} = \{\mathbf{m}_k\}_{k\geq 0}$ of degree $+1$

$$\mathbf{m}_k : B_k(C[1]) \to C[1], \quad \text{for } k = 0, 1, \cdots.$$

We note that $\mathbf{m}_0 : \Lambda_{0,nov} \to C[1]$. We assume that

$$\mathbf{m}_k \left(F^{\lambda_1}C^{m_1} \otimes \cdots \otimes F^{\lambda_k}C^{m_k}\right) \subseteq F^{\lambda_1+\cdots+\lambda_k}C^{m_1+\cdots+m_k-k+2} \tag{4.2.6}$$

and

$$\mathbf{m}_0(1) \in F^{\lambda'}C[1] \quad \text{for some } \lambda' > 0. \tag{4.2.7}$$

When we put

$$\overline{\mathbf{m}}_k(x_1, \ldots, x_k) \equiv \mathbf{m}_k(x_1, \ldots, x_k) \mod \Lambda^+_{0,nov} \tag{4.0}$$

for $k = 0, 1, 2, \ldots$, then $\{\overline{\mathbf{m}}_k\}$ defines an A_∞ algebra structure on $\overline{C} = C/\Lambda^+_{0,nov}C$ over $\mathbf{Q}$ introduced by Stasheff [St]. (Strictly speaking, he did not treat the map $\mathbf{m}_0$. But this map is important when we discuss the obstruction theory. See Remark 4.9. Note that the filtration is defined by λ which is the power of T. Thus on the $\overline{C}$, we do not have filtrations. In this way $(\overline{C}, \overline{\mathbf{m}} = \{\overline{\mathbf{m}}_k\}_{k\geq 0})$ becomes an A_∞ algebra over $\Lambda_{0,nov}/\Lambda^+_{0,nov} \cong \mathbf{Q}$.) We also assume that there exists a constant $\lambda'' > 0$ such that

$$\mathbf{m}_k(\mathbf{e}_{i_1}, \ldots, \mathbf{e}_{i_k}) - \overline{\mathbf{m}}_k(\mathbf{e}_{i_1}, \ldots, \mathbf{e}_{i_k}) \in F^{\lambda''}C[1]. \tag{4.2.8}$$

Here λ'' is independent of k and $\mathbf{e}_{i_1}, \ldots, \mathbf{e}_{i_k}$. The condition (4.2.8) is used when we construct a spectral sequence. See [FOOO] for details. We call (4.2.8) the *gap condition.*

Now the direct sum $B(C[1]) := \oplus_k B_k(C[1])$ has a structure of graded coalgebra. We regard $B(C[1])$ as a coalgebra and will construct a coderivation on it. The coproduct Δ is defined by :

$$\Delta(x_1 \otimes \cdots \otimes x_n) = \sum_{k=0}^{n} (x_1 \otimes \cdots \otimes x_k) \otimes (x_{k+1} \otimes \cdots \otimes x_n). \tag{4.3}$$

We can extend $\mathbf{m}_k$ uniquely to a coderivation

$$d_k : \bigoplus_n B_n(C[1]) \to \bigoplus_n B_{n-k+1}(C[1]),$$

by

$$d_k(x_1 \otimes \cdots \otimes x_n) = \sum_{\ell=1}^{n-k+1} (-1)^{\deg x_1 + \cdots + \deg x_{\ell-1} + \ell - 1} x_1 \otimes \cdots \otimes \mathbf{m}_k(x_\ell, \cdots, x_{\ell+k-1}) \otimes \cdots \otimes x_n \tag{4.4}$$

for $k \le n$ and $d_k = 0$ for $k > n$. Here and hereafter $\deg x$ means the degree of x *before* we shift it. When $k = 0$, we put $\mathbf{m}_0(1)$ in the right hand side. Namely we define d_0 by

$$d_0(x_1 \otimes \cdots \otimes x_n) = \sum_{\ell=1}^{n+1} (-1)^{\deg x_1 + \cdots + \deg x_{\ell-1} + \ell - 1} x_1 \otimes \cdots \otimes x_{\ell-1} \otimes \mathbf{m}_0(1) \otimes x_\ell \otimes \cdots \otimes x_n.$$

We want to consider the infinite sum $\widehat{d} = \sum d_k$. Therefore we need to consider a completion $\widehat{B}(C[1])$ of $B(C[1])$. We define a filtration $F^\lambda B_k(C[1])$ on $B_k(C[1])$ by

$$F^\lambda B_k(C[1]) = \bigcup_{\lambda_1 + \cdots + \lambda_k \ge \lambda} (F^{\lambda_1} C^{m_1} \otimes \cdots \otimes F^{\lambda_k} C^{m_k})$$

Let $\widehat{B}_k(C[1])$ be the completion with respect to the filtration.

Definition 4.5. $\widehat{B}(C[1])$ is the set of all formal sum $\sum_k \mathbf{x}_k$ where $\mathbf{x}_k \in \widehat{B}_k(C[1])$ such that

$$\mathbf{x}_k \in F^{\lambda_k} \widehat{B}_k(C[1])$$

with $\lim_{k\to\infty} \lambda_k \to \infty$.

Lemma 4.6. *If* (4.2) *is satisfied, then* $\widehat{d}$ *is well-defined as a map from* $\widehat{B}(C[1])$ *to* $\widehat{B}(C[1])$.

The proof is easy.

Now we introduce the following condition for an element $\mathbf{e}$ of C.

Condition 4.7.

(4.7.1) $\mathbf{m}_{k+1}(x_1, \cdots, \mathbf{e}, \cdots, x_k) = 0$, $k \ge 2$, $k = 0$.

(4.7.2) $\mathbf{m}_2(\mathbf{e}, x) = (-1)^{\deg x} \mathbf{m}_2(x, \mathbf{e}) = x$.

Definition 4.8. ([FOOO]) (1) $\mathbf{m} = \{\mathbf{m}_k\}_{k\ge 0}$ defines a structure of *filtered* A_∞ *algebra* on C if (4.2) are satisfied and if $\widehat{d} \circ \widehat{d} = 0$. We call $\widehat{B}(C[1])$ the (completed) *bar complex* associated to the A_∞ algebra $(C, \mathbf{m})$. If a filtered A_∞ algebra has an element $\mathbf{e}$ which satisfies Condition 4.7, the we call it *an* A_∞ *algebra with unit* and $\mathbf{e}$ a *unit.*

(2) For a filtered A_∞ algebra $(C, \mathbf{m})$, we say that a filtered A_∞ algebra $(C', \mathbf{m}')$ is *an* A_∞*-deformation* of $(C, \mathbf{m})$, if $(\overline{C}', \overline{\mathbf{m}}') = (\overline{C}, \overline{\mathbf{m}})$. Here $(\overline{C}', \overline{\mathbf{m}}')$ and $(\overline{C}, \overline{\mathbf{m}})$ are defined by reducing the coefficient ring $\Lambda_{0,nov}$ to

$\Lambda_{0,nov}/\Lambda_{0.nov}^{+} \cong \mathbf{Q}$. We also say that a filtered A_∞ algebra $(C, \mathbf{m})$ is *an A_∞-deformation* of an A_∞ algebra $(\overline{C}'', \overline{\mathbf{m}}'')$ if $(\overline{C}, \overline{\mathbf{m}}) = (\overline{C}'', \overline{\mathbf{m}}'')$.

Remark 4.9. (1) The equation $\widehat{d} \circ \widehat{d} = 0$ produces infinitely many relations among $\mathbf{m}_k$'s. For example, we have

$$\begin{gathered}
\mathbf{m}_1(\mathbf{m}_0(1)) = 0, \\
\mathbf{m}_2(\mathbf{m}_0(1), x) + (-1)^{\deg x+1}\mathbf{m}_2(x, \mathbf{m}_0(1)) + \mathbf{m}_1(\mathbf{m}_1(x)) = 0, \\
\mathbf{m}_3(\mathbf{m}_0(1), x, y) + (-1)^{\deg x+1}\mathbf{m}_3(x, \mathbf{m}_0(1), y) \\
+(-1)^{\deg x+\deg y+2}\mathbf{m}_3(x, y, \mathbf{m}_0(1)) \\
+\mathbf{m}_2(\mathbf{m}_1(x), y) + (-1)^{\deg x+1}\mathbf{m}_2(x, \mathbf{m}_1(y)) + \mathbf{m}_1(\mathbf{m}_2(x, y)) = 0, \\
\cdots\cdots\cdots
\end{gathered}$$

In general, it is easy to show that $\widehat{d} \circ \widehat{d} = 0$ is equivalent to that for each k

$$\sum_{k_1+k_2=k+1} \sum_i (-1)^{\deg x_1+\cdots+\deg x_{i-1}+i-1} \mathbf{m}_{k_1}(x_1, \cdots, \mathbf{m}_{k_2}(x_i, \cdots, x_{i+k_2-1}), \cdots, x_k) = 0.$$

If $\mathbf{m}_0 = 0$, then $\mathbf{m}_1\mathbf{m}_1 = 0$. So in this case $\mathbf{m}_1$ plays a role of a (co)boundary operator. In this sense, $\mathbf{m}_0$ describes an obstruction to that $\mathbf{m}_1\mathbf{m}_1 = 0$.

(2) In addition to $\mathbf{m}_0 = 0$, suppose that $\mathbf{m}_k = 0$ for $k \geq 3$. We put

$$\mathbf{m}_1(x) = (-1)^{\deg x}dx, \quad \text{and} \quad \mathbf{m}_2(x, y) = (-1)^{\deg x(\deg y+1)}x \wedge y,$$

where $\deg x$ denotes the degree of x as cochain. Then this is nothing but a DGA (differential graded algebra) and $\widehat{d} \circ \widehat{d} = 0$ implies that the usual Leibnitz rule and the associativity of the product structure. We note that the signs here are slightly different from those in [G-J]. (See also Remark 4.13 (2) below.)

4.B) A filtered A_∞ algebra associated to Lagrangian submanifold L.

Let $\overline{S}^k(L; \mathbf{Q})$ be a free $\mathbf{Q}$ module generated by all integral k-currents on L which are represented by singular chains. We denote by $C^k(L; \mathbf{Q})$ a countably generated submodule of $\overline{S}^k(L; \mathbf{Q})$. (We will use a method of "smooth correspondence". To do this we need and use the transversality argument and the Baire category theorem. This is why we introduce $C^k(L; \mathbf{Q})$. But the details are omitted here, see [FOOO].) Since an element in $C^k(L; \mathbf{Q})$ is represented by a singular chain, we sometimes write it as a singular chain representative (P, f). (But when we consider the orientation problems, we have to notice the difference of signs of boundary orientation and product (intersection) as *chain* or *cochain*, see Remark 4.13 (2) below.) We define $C^\bullet(L;\Lambda_{0,nov})$ by the completion of $C^\bullet(L;\mathbf{Q})\otimes\Lambda_{0,nov}$. For the convenience of notation, we put

$$C^\bullet = C^\bullet(L;\Lambda_{0,nov}) := (C^\bullet(L;\mathbf{Q})\otimes\Lambda_{0,nov})^\frown.$$

The degree in $C^\bullet(L;\Lambda_{0,nov})$ is the sum of the degree in $C^\bullet(L;\mathbf{Q})$ and the degree of the coefficients in $\Lambda_{0,nov}$. Using the filtration on $\Lambda_{0,nov}$, we can uniquely define the filtration on $C^k(L;\Lambda_{0,nov})$ which satisfies the following conditions;

$$C^k(L;\mathbf{Q}) \subset F^0C^k(L;\Lambda_{0,nov})$$

and

$$C^k(L;\mathbf{Q}) \not\subset F^\lambda C^k(L;\Lambda_{0,nov}) \quad \text{for } \lambda>0.$$

We now define the maps

$$\mathbf{m}_k : B_k(C[1](L;\Lambda_{0,nov})) \to C[1](L;\Lambda_{0,nov}).$$

of degree $+1$ for $k\ge 0$. To do this, we recall that $\mathcal{M}_{k+1}(\beta)$ is the set of pairs $((\Sigma,\vec{z}),w)$ where $w:(\Sigma,\partial\Sigma)\to(M,L)$ is a pseudoholomorphic map which represents the class β. Let $\mathcal{M}^{\mathrm{main}}_{k+1}(\beta)$ be the subset of $\mathcal{M}_{k+1}(\beta)$ consisting of elements $((\Sigma,\vec{z}),w)$ where the order of the marked points is cyclic. (See Definition 2.1). For given

$$(P_i,f_i)\in C^{g_i}(L;\mathbf{Q}),\quad i=1,\cdots,k,$$

we consider

$$\mathcal{M}^{\mathrm{main}}_{k+1}(\beta)_{(ev_1,\cdots,ev_k)}\times_{f_1\times\cdots\times f_k}(P_1\times\cdots\times P_k).$$

Proposition 4.10. *Suppose L is relatively spin. Then*

$$\mathcal{M}^{\mathrm{main}}_{k+1}(\beta)_{(ev_1,\cdots,ev_k)}\times_{f_1\times\cdots\times f_k}(P_1\times\cdots\times P_k)$$

has an oriented Kuranishi structure. Its dimension is $n-\sum(g_i-1)+\mu(\beta)-2$, where $n=\dim L$.

As in Definition 3.1, we define $\mathcal{M}^{\mathrm{main}}_1(\beta;P_1,\cdots,P_k)$ by the following.

Definition 4.11.

$$\mathcal{M}^{\mathrm{main}}_1(\beta;P_1,\cdots,P_k) := (-1)^\epsilon\mathcal{M}^{\mathrm{main}}_{k+1}(\beta)_{(ev_1,\cdots,ev_k)}\times_{f_1\times\cdots\times f_k}(P_1\times\cdots\times P_k),$$

$$\epsilon = (n+1)\sum_{j=1}^{k-1}\sum_{i=1}^{j}\deg P_i.$$

Now we define the maps $\mathbf{m}_k$. We recall that we have the element $\beta_0 = 0 \in G_{++}(L)$ which satisfies $\mu(\beta_0) = 0$ and $\mathcal{A}(\beta_0) = 0$.

Definition 4.12. (1) For $(P, f) \in C^k(L, \mathbf{Q})$, we define

$$\mathbf{m}_{0,\beta}(1) = \begin{cases} (\mathcal{M}_1(\beta), ev_0) & \text{for } \beta \neq \beta_0 \\ 0 & \text{for } \beta = \beta_0, \end{cases}$$

$$\mathbf{m}_{1,\beta}(P, f) = \begin{cases} (\mathcal{M}_1^{\text{main}}(\beta; P), ev_0) & \text{for } \beta \neq \beta_0 \\ (-1)^n \partial P & \text{for } \beta = \beta_0, \end{cases}$$

and

$$\mathbf{e} = [L] \quad \text{(the fundamental cycle).}$$

The notation ∂ in the definition of $\mathbf{m}_{1,\beta_0}$ is the usual (classical) boundary operator.

(2) For each $k \geq 2$ and $(P_i, f_i) \in C^{g_i}(L, \mathbf{Q})$, we define $\mathbf{m}_{k,\beta}$ by

$$\begin{aligned} \mathbf{m}_{k,\beta}((P_1, f_1), \ldots, (P_k, f_k)) &= \mathbf{m}_{k,\beta}((P_1, f_1) \otimes \cdots \otimes (P_k, f_k)) \\ &= (\mathcal{M}_1^{\text{main}}(\beta; P_1, \cdots, P_k), ev_0). \end{aligned}$$

(3) Then we define $\mathbf{m}_k$ $(k \geq 0)$ by

$$\mathbf{m}_k = \sum_{\beta \in \pi_2(M,L)} \mathbf{m}_{k,\beta} \otimes [\beta] = \sum_{\beta \in \pi_2(M,L)} \mathbf{m}_{k,\beta} \otimes T^{\omega(\beta)} e^{\frac{\mu(\beta)}{2}}.$$

Remark 4.13. (1) By definition, $\overline{\mathbf{m}}_0 = 0$. (This is the case corresponding to $\beta = \beta_0$.) But $\mathbf{m}_0 \neq 0$.

(2) In the definition of $\mathbf{m}_{1,\beta_0}$ above, we see P as a *chain*. If we see P as a *cochain* (or a differential form), then we have

$$\mathbf{m}_{1,\beta_0}(P) = (-1)^{n+\deg P+1} dP, \tag{4.13.2}$$

where $\deg P$ is the degree of P as a cochain. This is because we can see the following general formula (under certain our conventions [FOOO] about orientations of boundary and of normal bundle). For an s-dim chain S in L, we have

$$P.D.(\partial S) = (-1)^{\deg S+1} d(P.D.(S)).$$

Here $\deg S = n - s$ and $P.D.$ denotes the Poincaré duality. Of course, this sign depends on a convention about the Poincaré duality. Actually, we use the following convention. For a chain S in L, the Poincaré dual $P.D.(S)$ satisfies

$$\int_S \alpha|_S = \int_L P.D.(S) \wedge \alpha$$

for any $\alpha \in \Omega^{\dim S}(L)$. We also note that the universal constant $n+1$ in the power of the sign in (4.13.2) does not affect in the A_∞ relations in the case $\mathbf{m}_0 = 0$. In this sense, this is consistent with Remark 4.9 (2).

By using the $\{\mathbf{m}_k\}_{k\geq 0}$, we define

$$d_k : \bigoplus_n B_n(C[1](L;\Lambda_{0,nov})) \to \bigoplus_n B_{n-k+1}(C[1](L;\Lambda_{0,nov}))$$

as in (4.4). Then the following is our main theorem in this section.

Theorem 4.14. ([FOOO]) *Suppose L is a relatively spin Lagrangian submanifold. Then $(C(L;\Lambda_{0,nov}),\mathbf{m})$ is a filtered A_∞ algebra (with unit $\mathbf{e}$). Furthermore, $(C(L;\Lambda_{0,nov}),\mathbf{m})$ satisfies the gap condition (4.2.8).*

(Strictly speaking, $\mathbf{e}$ is not a unit, but a *homotopy unit.* (We can deform (unitarize) $\mathbf{e}$ to be unit.) But we omit the details, see [FOOO].) Moreover, by using moduli space of *metric ribbon trees*, we can construct an A_∞ algebra $(\mathbf{Q}\mathcal{X},\overline{\mathbf{m}})$ over $\mathbf{Q}$ with $\overline{\mathbf{m}}_0 = 0$, such that it describes the rational homotopy type of L and the cohomology of $\overline{\mathbf{m}}_1$ is isomorphic to the cohomology of L [FOOO]. Then we can show the following.

Theorem 4.15. ([FOOO]) *$(C(L;\Lambda_{0,nov}),\mathbf{m})$ is an A_∞ deformation of the A_∞ algebra $(\mathbf{Q}\mathcal{X},\overline{\mathbf{m}})$.*

Sketch of the Proof of Theorem 4.14: To prove $\widehat{d}\circ\widehat{d} = 0$, we analyze the boundary of $\mathcal{M}_1^{\mathrm{main}}(\beta;P_1,\cdots,P_k)$. We find that its boundary is the sum of

$$\sum_i \mathcal{M}_1^{\mathrm{main}}(\beta;P_1,\cdots,\partial P_i,\cdots,P_k)$$

and the terms described by the bubbling off holomorphic discs. On the other hand, in order to prove $\widehat{d}\circ\widehat{d} = 0$, we note that it is enough to show that

$$(4.16.1)\quad \sum_{\beta_1+\beta_2=\beta}\ \sum_{k_1+k_2=k+1}\ \sum_i (-1)^{\deg P_1+\cdots+\deg P_{i-1}+i-1} \mathbf{m}_{k_1,\beta_1}(P_1,\cdots,\mathbf{m}_{k_2,\beta_2}(P_i,\cdots,P_{i+k_2-1}),\cdots,P_k) = 0.$$

(See Remark 4.9.) We divide the left hand side into 3 terms, according as $\beta_1 = 0 (= \beta_0)$ and $k_1 = 1$, $\beta_2 = 0(= \beta_0)$ and $k_2 = 1$, and the other cases. Then we can rewrite the left hand side in (4.16.1) as follows:

$$
\begin{aligned}
&\mathbf{m}_{1,0}\mathbf{m}_{k,\beta}(P_1,\cdots,P_k)\\
(4.16.2)\qquad &+\sum_i(-1)^{\deg P_1+\cdots+\deg P_{i-1}+i-1}\mathbf{m}_{k,\beta}(P_1,\cdots,\mathbf{m}_{1,0}(P_i),\cdots,P_k)\\
&+\sum_{\substack{\beta_1+\beta_2=\beta,\ k_1+k_2=k+1;\\ \beta_1\neq 0 \text{ or } k_1\neq 1,\\ \beta_2\neq 0 \text{ or } k_2\neq 1}}\sum_i(-1)^{\deg P_1+\cdots+\deg P_{i-1}+i-1}\\
&\qquad\mathbf{m}_{k_1,\beta_1}(P_1,\cdots,\mathbf{m}_{k_2,\beta_2}(P_i,\cdots,P_{i+k_2-1}),\cdots,P_k).
\end{aligned}
$$

By Definition 4.12, we have $\mathbf{m}_{1,0} = (-1)^n\partial$, where ∂ is the classical boundary map. Hence the first term in (4.16.2) is nothing but

$$(4.16.2.1)\qquad (-1)^n\partial(\mathcal{M}_1^{\mathrm{main}}(\beta : P_1,\cdots,P_k), ev_0),$$

and the second term in (4.16.2) is the sum of

$$(4.16.2.2)\qquad (-1)^{\sum_{j=1}^{i-1}(\deg P_j+1)}(-1)^n(\mathcal{M}_1^{\mathrm{main}}(\beta : P_1,\cdots,\partial P_i,\cdots,P_k), ev_0).$$

The third term in (4.16.2) geometrically corresponds to moduli spaces described by bubbling off holomorphic discs. This is the sum of

$$
\begin{aligned}
(4.16.2.3)\qquad &(-1)^{\sum_{j=1}^{i-1}(\deg P_j+1)}(\mathcal{M}_1^{\mathrm{main}}(\beta_1; P_1,\cdots\\
&\cdots,P_{i-1},\mathcal{M}_1^{\mathrm{main}}(\beta_2; P_i,\cdots,P_{i+k_2-1}),\cdots,P_k), ev_0).
\end{aligned}
$$

Moreover, as for the orientations of these spaces, we can show the following:

$$
\begin{aligned}
(4.16.3.1)\qquad &(-1)^{\sum_{j=1}^{i-1}(\deg P_j+1)}(-1)^n\mathcal{M}_1^{\mathrm{main}}(\beta : P_1,\cdots,\partial P_i,\cdots,P_k)\\
&\subseteq (-1)^{n+1}\partial\mathcal{M}_1^{\mathrm{main}}(\beta; P_1,\cdots,P_k)
\end{aligned}
$$

and

(4.16.3.2)

$$
\begin{aligned}
&(-1)^{\sum_{j=1}^{i-1}(\deg P_j+1)}\mathcal{M}_1^{\mathrm{main}}(\beta_1; P_1,\cdots,\mathcal{M}_1^{\mathrm{main}}(\beta_2; P_i,\cdots,P_{i+k_2-1}),\cdots,P_k)\\
&\subseteq (-1)^{n+1}\partial\mathcal{M}_1^{\mathrm{main}}(\beta; P_1,\cdots,P_k).
\end{aligned}
$$

Therefore we find that (4.16.2.1) and the sum of (4.16.2.2) and (4.16.2.3) cancel each other. Namely (4.16.2) is zero. This implies $\widehat{d}\circ\widehat{d} = 0$. We recall that we have an element $\beta_1 \in \pi_2(M, L)$ such that it is represented by J holomorphic disc with the minimal (non zero) area. We take $\lambda'' > 0$ such that

$$\lambda'' < \omega[\beta_1].$$

Then we can find that $\{\mathbf{m}_k\}$ satisfies the gap condition (4.2.8).

§5. Bounding cochains and deformation.

From now on, we are working on cohains (or cohomologies), not on chains (or homologies) via the Poincaré duality, because they are fitted with the framework of obstruction theory.

5.A) Bounding cochains and the master equation.

For a cochain $b \in C[1]^0(L, \Lambda_{0,nov})$ with the shifted degree 0, we put

$$e^b = 1 + b + b \otimes b + b \otimes b \otimes b + \cdots \in \widehat{B}(C[1](L, \Lambda_{0,nov}). \tag{5.1}$$

(We do not put the factorials here unlike definition of the exponential, because we use only the main component among $(k+1)!$ components of $\mathcal{M}_{k+1}$ to define the map $\mathbf{m}_k$.)

Definition 5.2. We say that b is a *bounding cochain* if $\widehat{d}e^b = 0$. A filtered A_∞ algebra is said to be *unobstructed* if there exists a bounding cochain and *obstructed* otherwise. Similarly, we call a Lagrangian submanifold L *unobstructed* if the associated filtered A_∞ algebra $(C(L; \Lambda_{0,nov}), \mathbf{m})$ constructed in Section 4.B) is unobstructed. We denote by $\widehat{\mathcal{M}}(L) = \widehat{\mathcal{M}}(L; J, \Xi)$ the set of all bounding cochains b. Here J stands for a compatible almost complex structure and Ξ for a parameter of the Kuranishi structure.

From the construction in §3, we put

$$b = \sum \mathcal{B}(\beta_i) \otimes [\beta_i] = \sum \mathcal{B}(\beta_i) \otimes T^{\omega(\beta_i)} e^{\frac{\mu(\beta_i)}{2}} \in C[1]^0(L; \Lambda_{0,nov}). \tag{5.3}$$

Then we can show that

Lemma 5.4. *The chains $\mathcal{B}(\beta_i)$ bound $o_i(L)$ inductively if and only if b in* (5.3) *is a bounding cochain, i.e.,* $\widehat{d}e^b = 0$.

Remark 5.5. The equation $\widehat{d}e^b = 0$ is equivalent to

$$\mathbf{m}_0(1) + \mathbf{m}_1(b) + \mathbf{m}_2(b, b) + \mathbf{m}_3(b, b, b) + \cdots = 0.$$

If $\mathbf{m}_0 = 0$ and $\mathbf{m}_k = 0$ for $k \geq 3$, by putting $\mathbf{m}_1 = d$ and $\mathbf{m}_2 = \wedge$, the equation is equivalent to

$$db + b \wedge b = 0,$$

which is nothing but the classical Maurer-Cartan equation for DGA. Our equation $\widehat{d}e^b = 0$ is an inhomogeneous A_∞-version of Maurer-Cartan or Batalin-Vilkovisky master equation [BV]. The relation of Batalin-Vilkovisky master equation to the deformation theory is discussed in [Sch], [ASKZ], [BK], [K].

The deformation of the Floer coboundary operators in §3 can be interpreted as follows. Here for simplicity, we discuss the case for one Lagrangian submanifold L. (The case for two Lagrangian submanifolds L_0 and L_1 is similar, but needs more notations and argument, e.g., we have to use Λ_{nov}.) Suppose that the filtered A_∞ algebra $(C(L;\Lambda_{0,nov}),\mathbf{m})$ we constructed in §4 is unobstructed in the sense of Definition 5.2. Then we have bounding cochains $b_1, b_2 \in C[1]^0(L,\Lambda_{0,nov})$. (They may coincide.) By using these cochains b_1, b_2, we define

$$\delta_{b_1,b_2} : C(L;\Lambda_{0,nov}) \to C(L;\Lambda_{0,nov})$$

by

$$\delta_{b_1,b_2}(x) = \sum_{k_1,k_2\ge 0} \mathbf{m}_{k_1+k_2+1}(\underbrace{b_1,\cdots,b_1}_{k_1},x,\underbrace{b_2,\cdots,b_2}_{k_2}). \tag{5.6}$$

Then we can find that

$$\widehat{d}(e^{b_1}xe^{b_2}) = e^{b_1}\delta_{b_1,b_2}(x)e^{b_2} + \widehat{d}(e^{b_1})xe^{b_2} + (-1)^{\deg x+1}e^{b_1}x\widehat{d}(e^{b_2}).$$

The second and the third term vanishes if $\widehat{d}(e^{b_1}) = \widehat{d}(e^{b_2}) = 0$. Thus we have

Proposition 5.7. *If $\widehat{d}(e^{b_1}) = \widehat{d}(e^{b_2}) = 0$, then $\delta_{b_1,b_2}\circ\delta_{b_1,b_2} = 0$.*

5.B) Deformation of A_∞ algebra.

Now let $b \in C[1]^0(L,\Lambda_{0,nov})$ be a cochain, which is not necessary a bounding cochain. For the cochain b, we next deform our filtered A_∞ algebra as follows.

Definition 5.8. Using this cochain b, we put

$$\begin{aligned}\mathbf{m}_k^b(x_1,\cdots,x_k) &= \sum_{\ell_0,\cdots,\ell_k} \mathbf{m}_{k+\sum\ell_i}(\underbrace{b,\cdots,b}_{\ell_0},x_1,\underbrace{b,\cdots,b}_{\ell_1},\cdots,\underbrace{b,\cdots,b}_{\ell_{k-1}},x_k,\underbrace{b,\cdots,b}_{\ell_k})\\ &= \mathbf{m}(e^bx_1e^bx_2\cdots x_{k-1}e^bx_ke^b)\end{aligned}$$

for $k = 0,1,2\cdots$.

We note that $\mathbf{m}_0^b(1) = \mathbf{m}(e^b)$. Since we can find that

$$\widehat{d}e^b = e^b\mathbf{m}_0^b(1)e^b,$$

we have the following.

Proposition 5.9. *$(C, \mathbf{m}^b)$ is also a filtered A_∞ algebra. In addition, $\widehat{d}(e^b) = 0$ is equivalent to $\mathbf{m}_0^b = 0$.*

This implies that an unobstructed filtered A_∞ algebra can be deformed to a filtered A_∞ algebra with $\mathbf{m}_0 = 0$.

5.C) Homotopy equivalence, dependence and independence.

Let $(C_i, \mathbf{m}^i)$, $i = 1, 2$, be filtered A_∞ algebras over the ring $\Lambda_{0,nov}$. For $k = 0, 1, 2, \cdots$, let us consider the family of maps

$$\mathbf{f}_k : B_k(C_1[1]) \to C_2[1]$$

of degree 0 such that

$$\mathbf{f}_k(F^\lambda B_k(C_1[1])) \subseteq F^\lambda C_2[1] \tag{5.10.1}$$

and

$$\mathbf{f}_0(1) \in F^{\lambda'} C_2[1] \quad \text{for some } \lambda' > 0. \tag{5.10.2}$$

Note that $\mathbf{f}_0 : \Lambda_{0,nov} \to C_2[1]$. These maps induce

$$\varphi_k : B_k(C_1[1]) \to \widehat{B}(C_2[1]),$$

by

$$\begin{aligned} \varphi_k(x_1 \otimes \cdots \otimes x_k) = \sum_{0 \le k_1 \le \cdots \le k_n \le k} & \mathbf{f}_{k_1}(x_1, \cdots, x_{k_1}) \otimes \cdots \\ & \cdots \otimes \mathbf{f}_{k_{i+1}-k_i}(x_{k_i+1}, \cdots, x_{k_{i+1}}) \otimes \cdots \\ & \cdots \otimes \mathbf{f}_{k-k_n}(x_{k_n+1}, \cdots, x_k), \end{aligned} \tag{5.11}$$

and (5.10) implies that $\sum \varphi_k = \widehat{\varphi}$ converges. We note that when $\mathbf{f}_0$ appears in the right hand side of (5.11), we put $\mathbf{f}_0(1)$ there. Thus, in particular, φ_0 is given by

$$\varphi_0(1) = 1 + \mathbf{f}_0(1) + \mathbf{f}_0(1) \otimes \mathbf{f}_0(1) + \cdots = e^{\mathbf{f}_0(1)}$$

in our notation. (See (5.1)). Then it is easy to see that $\widehat{\varphi} : \widehat{B}(C_1[1]) \to \widehat{B}(C_2[1])$ is a coalgebra homomorphism.

Definition 5.12. We call $\mathbf{f} = \{\mathbf{f}_k\}_{k \ge 0}$ a *filtered A_∞ homomorphism from C_1 to C_2* if $\widehat{\varphi} \circ \widehat{d}^1 = \widehat{d}^2 \circ \widehat{\varphi}$.

Let $\mathbf{e}_{i_j}$ be a basis of C_1 as in (4.2.5). We say that $\mathbf{f}$ satisfies the *gap condition* if

$$\mathbf{f}_k(\mathbf{e}_{i_1},\ldots,\mathbf{e}_{i_k}) - \overline{\mathbf{f}}_k(\mathbf{e}_{i_1},\ldots,\mathbf{e}_{i_k}) \in F^{\lambda''}C_2[1] \tag{5.12.1}$$

where $\lambda'' > 0$ is independent of i_j and k. (Here $\overline{\mathbf{f}}$ denotes the induced map on the (not filtered) A_∞ algebra over $\Lambda_{0,nov}/\Lambda^+_{0,nov} \cong \mathbf{Q}$, (see §4.))

Let $\mathbf{f}^i_k : B_k(C_i[1]) \to C_{i+1}[1]$ $(i = 1,2)$ define a filtered A_∞ homomorphism. Then the composition $\mathbf{f}^2 \circ \mathbf{f}^1 = \{(\mathbf{f}^2 \circ \mathbf{f}^1)_k\}$ of $\mathbf{f}^1$ and $\mathbf{f}^2$ is

$$\begin{aligned}&(\mathbf{f}^2 \circ \mathbf{f}^1)_k(x_1,\cdots,x_k)\\ &= \sum_m \sum_{k_1+\cdots+k_m=k} \mathbf{f}^2_m(\mathbf{f}^1_{k_1}(x_1,\cdots,x_{k_1}),\cdots,\mathbf{f}^1_{k_m}(x_{k-k_m+1},\cdots,x_k)).\end{aligned}$$

which defines a filtered A_∞ homomorphism from C_1 to C_3.

Let $(C_i,\mathbf{m}^i)$ $(i = 1,2)$ be filtered A_∞ algebras over $\Lambda_{0,nov}$ and $\mathbf{f} : (C_1,\mathbf{m}^1) \to (C_2,\mathbf{m}^2)$ a filtered A_∞ homomorphism. Then $\mathbf{f}$ naturally induces an A_∞ homomorphism $\overline{\mathbf{f}} : (\overline{C}_1,\overline{\mathbf{m}}^1) \to (\overline{C}_2,\overline{\mathbf{m}}^2)$, where $(\overline{C}_i,\overline{\mathbf{m}}^i)$ are the A_∞ algebras over $\mathbf{Q} \cong \Lambda_{0,nov}/\Lambda^+_{0,nov}$. If $\overline{\mathbf{m}}_0 = 0$, then we note that $\overline{\mathbf{m}}_1\overline{\mathbf{m}}_1 = 0$, (see Remark 4.9).

Definition 5.13. Let $(C_i,\mathbf{m}^i)$ $(i = 1,2)$ be filtered A_∞ algebras over $\Lambda_{0,nov}$ such that $\overline{\mathbf{m}}^i_0 = 0$. For these filtered A_∞ algebras, we say that a filtered A_∞ homomorphism $\mathbf{f} : (C_1,\mathbf{m}^1) \to (C_2,\mathbf{m}^2)$ is a *weak homotopy equivalence*, if the induced A_∞ homomorphism $\overline{\mathbf{f}} : (\overline{C}_1,\overline{\mathbf{m}}^1) \to (\overline{C}_2,\overline{\mathbf{m}}^2)$ induces an isomorphism $\overline{\mathbf{f}}_1 : H^*(\overline{C}_1,\overline{\mathbf{m}}^1_1) \to H^*(\overline{C}_2,\overline{\mathbf{m}}^2_1)$.

We recall that the condition $\overline{\mathbf{m}}_0 = 0$ is satisfied in our filtered A_∞ algebra $(C(L,\Lambda_{0,nov}),\mathbf{m})$, see Definition 4.12.

Hereafter we assume that the filtered A_∞ algebras $(C_i,\mathbf{m}^i)$ are *unobstructed* and *weakly finite*. Here, a filtered A_∞ algebra $(C,\mathbf{m})$ is called *unobstructed* if there exists a bounding cochain $b \in C[1]^0$ such that $\widehat{d}(e^b) = 0$, and *weakly finite* if there exists a finite $\Lambda_{0,nov}$ module cochain complex (C',δ') such that there is a filtered A_∞ homomorphism $\mathbf{f}' : (C',\delta') \to (C,\mathbf{m})$ with satisfying the gap condition (5.12.1) which induces an isomorphism between $H^*(C',\delta')$ and $H^*(C[1],\mathbf{m}^b_1)$, (see Definition 5.8 and Proposition 5.9 for $\mathbf{m}^b$. We also note that our unobstructed filtered A_∞ algebra $(C(L,\Lambda_{0,nov}),\mathbf{m})$ associated to Lagrangian submanifold L is weakly finite, see [FOOO] Theorem A4.28 in §A4). Under these assumptions, we can obtain the following lemma. (Kontsevich shows a similar lemma in the case of L_∞ algebra [K]).

Lemma 5.14. *Let $(C_i, \mathbf{m}^i)$ be unobstructed and weakly finite filtered A_∞ algebras $(i = 1, 2)$. If a filtered A_∞ homomorphism $\mathbf{f}^1$ is a weak homotopy equivalence and if it satisfies the gap condition (5.12.1), then there exists a filtered A_∞ homomorphism $\mathbf{f}^2$ such that both of the compositions $(\mathbf{f}^1 \circ \mathbf{f}^2)_1$ and $(\mathbf{f}^2 \circ \mathbf{f}^1)_1$ induce the identities on the cohomologies $H^*(\overline{C}_i[1],\ \overline{\mathbf{m}}_1^i)$.*

Definition 5.15. Let $(C_i, \mathbf{m}^i)$ be filtered A_∞ algebras over $\Lambda_{0,nov}$ with $\overline{\mathbf{m}}_0^i = 0$ $(i = 1, 2)$. We assume that $(C_i, \mathbf{m}^i)$ are unobstructed and weakly finite. Then $(C_1, \mathbf{m}^1)$ and $(C_2, \mathbf{m}^2)$ are said to be *weakly homotopy equivalent* if there exist filtered A_∞ homomorphisms $\mathbf{f}^1$ and $\mathbf{f}^2$ from C_1 to C_2 and C_2 to C_1 respectively such that the compositions $\mathbf{f}^1 \circ \mathbf{f}^2$, $\mathbf{f}^2 \circ \mathbf{f}^1$ are weakly homotopy equivalences.

Now we recall from Proposition 5.7 that two bounding cochains b_1, b_2 on filtered A_∞ algebra $(C, \mathbf{m})$ induce a coboundary map

$$\delta_{b_1,b_2} : C \to C$$

as in (5.6). We next prove that a weak homotopy equivalence induces a natural isomorphism between the cohomology of (C, δ_{b_1,b_2}). We first note the following lemma.

Lemma 5.16. *For a non-zero element $\mathbf{x}$ of $\widehat{B}(C[1])$, $\mathbf{x} = e^b$ for some $b \in B_1(C[1]) = C[1]$ if and only if $\Delta\mathbf{x} = \mathbf{x} \otimes \mathbf{x}$, where Δ is the coproduct as in (4.3).*

Since $\widehat{\varphi} : \widehat{B}(C[1]) \to \widehat{B}(C[1])$ is a coalgebra homomorphism, if $\mathbf{x}$ satisfies $\Delta\mathbf{x} = \mathbf{x} \otimes \mathbf{x}$, so does $\widehat{\varphi}(\mathbf{x})$ and so we have an element $\varphi(b_0)$ such that $\widehat{\varphi}(e^{b_0}) = e^{\varphi(b_0)}$ by Lemma 5.16. More explicitly, we have

$$\varphi(b_0) = \mathbf{f}_0(1) + \mathbf{f}_1(b_0) + \mathbf{f}_2(b_0, b_0) + \cdots. \tag{5.17}$$

Proposition 5.18. *Let $(C_0, \mathbf{m}^0)$ and $(C_1, \mathbf{m}^1)$ be the filtered A_∞ algebras such that $\overline{\mathbf{m}}_0^i = 0$ $(i = 0, 1)$. We assume that $(C_i, \mathbf{m}^i)$ are unobstructed and weakly finite. Let $\mathbf{f} = \{\mathbf{f}_k\}_{k \geq 0}$ define a weak homotopy equivalence between them. Let $\widehat{d^0}$, $\widehat{d^1}$ be obtained from $\mathbf{m}^0$, $\mathbf{m}^1$, and φ obtained from $\mathbf{f}_k$. Let $b_0 \in (C_0[1])^0$. Then $\widehat{d^0} e^{b_0} = 0$ if and only if $\widehat{d^1} e^{\varphi(b_0)} = 0$.*

If we moreover assume the gap condition for $(C_0, \mathbf{m}^0)$, $(C_1, \mathbf{m}^1)$ and $\mathbf{f}$, then the cohomology of $\delta^0_{b_1,b_2}$ is isomorphic to that of $\delta^1_{\varphi(b_1),\varphi(b_2)}$, $\widehat{d^0} e^{b_0} = 0$.

For the proof of the last assertion, we need a spectral sequence argument. To construct the spectral sequence, we need the gap condition. See [FOOO] for more details.

Next, we can define an equivalence relation $\sim$ in the set of bounding cochains $\widehat{\mathcal{M}}(L)$, (see Definition 5.2). This can be regarded as a sort of "gauge equivalence" relation. A similar notion is introduced in [K] for L_∞ algebra. We do not explain it here. See [FOOO]. Anyway, we can show that the following.

Theorem 5.19. *Let (L, L') be a pair of relative spin Lagrangian submanifolds. Let $b_0, b_1 \in \widehat{\mathcal{M}}(L)$ and $b'_0, b'_1 \in \widehat{\mathcal{M}}(L')$. Assume that $b_0 \sim b_1$ and $b'_0 \sim b'_1$. Then the deformed Floer cohomology $HF((L, b_0), (L', b'_0))$ is canonically isomorphic to $HF((L, b_1), (L', b'_1))$*

More generally, we can show the followings. We set $\mathcal{M}(L) = \widehat{\mathcal{M}}(L)/\sim$.

Theorem 5.20. *Let (L, L') be a pair of relative spin Lagrangian submanifolds of M. Then we have the following:*

(5.20.1) *$\mathcal{M}(L; J, \Xi)$ is independent of the choice of J, Ξ. Namely there exists a canonical isomorphism $\mathcal{M}(L; J, \Xi) \cong \mathcal{M}(L; J', \Xi')$. (Hereafter we omit J, Ξ and write $\mathcal{M}(L)$ in case no confusion can occur.)*

(5.20.2) *Floer cohomology is also independent of J, Ξ. More precisely we have the following : Let $b_0 \in \mathcal{M}(L; J_0, \Xi_0)$, $b'_0 \in \mathcal{M}(L'; J_0, \Xi'_0)$. Let $b_1 \in \mathcal{M}(L; J_1, \Xi_1)$, and $b'_1 \in \mathcal{M}(L'; J_1, \Xi'_1)$ corresponds to them by the isomorphism in* (5.20.1). *Then there exists a canonical isomorphism*

$$HF((L, b_0), (L', b'_0); J_0, \Xi_0, \Xi'_0) \cong HF((L, b_1), (L', b'_1); J_1, \Xi_1, \Xi'_1).$$

Hereafter we write $HF((L, b_0), (L', b'_0))$ in place of

$$HF((L, b_0), (L', b'_0); J_0, \Xi_0, \Xi'_0)$$

when no confusion can occur.

(5.20.3) *Any Hamiltonian diffeomorphism ψ induces a map*

$$\psi_* : \mathcal{M}(L) \simeq \mathcal{M}(\psi(L)),$$

which depends only on the homotopy class of the Hamiltonian diffeomorphism $\psi : L \to \psi(L)$. Namely if ψ^s be a family of Hamiltonian diffeomorphisms such that $\psi^s(L)$ is independent of s then $\psi^0_* = \psi^1_*$.

(5.20.4) *Let $\Psi = \{\psi^\tau\}_{0 \le \tau \le 1}$ and $\Psi' = \{\psi'^\tau\}_{0 \le \tau \le 1}$ be Hamiltonian isotopies with $\psi^0 = \psi'^0 = id$. Let $b_0 \in \mathcal{M}(L)$, and $b'_0 \in \mathcal{M}(L')$. We put $b_1 = \psi^1_*(b_0) \in \mathcal{M}(\psi^1(L), J_1)$, $\psi'^1_*(b'_0) = b'_1 \in \mathcal{M}(\psi'^1(L'))$. Then Ψ, Ψ' induces an isomorphism*

$$(\Psi, \Psi')_* : HF((L, b_0), (L', b'_0)) \cong HF((\psi^1(L), b_1), (\psi'^1(L'), b'_1)),$$

which depends only on homotopy types of Hamiltonian isotopies between id and ψ^1 and between id and ψ'^1

Theorem 5.20 says, up to ambiguity of the choice of $\mathcal{B}$, the obstruction class and Floer cohomology are independent of the Hamiltonian isotopy and of the almost complex structure.

Moreover, we can show that $\mathcal{M}(L)$ can be described as some quotient space of the zero set of certain formal map (Kuranishi map). So it describes the deformation space. See [FOOO].

§6. Some applications.

In this last section, we give some applications of our theory to some concrete problems in symplectic geometry. For the proofs, see [FOOO].

The first one is the Arnold conjecture for Lagrangian intersections.

Theorem 6.1. *Assume that L is relatively spin closed Lagrangian submanifold of (M, ω) and that the natural map $H_*(L; \mathbf{Q}) \to H_*(M; \mathbf{Q})$ is injective. Then for any Hamiltonian diffeomorphism $\psi : M \to M$ such that L and $\psi(L)$ intersect transversally, we have*

$$\sharp(L \cap \psi L) \geq \sum_k \mathrm{rank} H_k(L; \mathbf{Q}).$$

The assumption that the natural map $H_*(L; \mathbf{Q}) \to H_*(M; \mathbf{Q})$ is injective implies that all our obstruction classes vanish. (See Theorem 2.6). We remark that this theorem implies the Arnold conjecture for the fixed point sets of Hamiltonian diffeomorphisms (over $\mathbf{Q}$-coefficients) which is proved by [FO], [LT] etc. Namely, let us consider $L = \Delta$ (the diagonal set) in $(M \times M, \omega \oplus -\omega)$. Then the intersection points are nothing but the fixed points of ψ. The relative spinness for Δ and the assumption above are automatically satisfied by the Kunneth formula.

More generally, by using our spectral sequence, we can get the following.

Theorem 6.2. *Let L be relatively spin and assume that the associated A_∞ algebra is unobstructed. Denote $A = \sum \mathrm{rank}\ H(L; \mathbf{Q})$ and*

$$B = \sum \mathrm{rank}\ker(H(L; \mathbf{Q}) \to H(M; \mathbf{Q})).$$

Then we have

$$\sharp(L \cap \psi(L)) \geq A - 2B$$

for any Hamiltonian diffeomorphism $\psi : M \to M$ such that L and $\psi(L)$ intersect transversally.

Next application is so called Arnold-Givental conjecture, which is a variant of Arnold conjecture. In general, the most naive statement such as

$$\sharp(L \cap \phi(L)) \geq \sum \mathrm{rank} H_*(L; \mathbf{Z}/2\mathbf{Z}) \tag{6.3}$$

is *not* true for general L and general Hamiltonian diffeomorphism ϕ. In this respect, Givental made a conjecture that (6.3) is true at least if L is the fixed point set of an anti-symplectic involution. However a careful analysis on the orientation of the moduli space shows that this cancellation does *not* happen over $\mathbf{Q}$ (or over $\mathbf{Z}$) but works only over $\mathbf{Z}/2\mathbf{Z}$-coefficient in general. Now we can prove the following :

Theorem 6.4. *Let $L = \mathrm{Fix}\,\tau$ be the fixed point set of an anti-symplectic involution $\tau : (M,\omega) \to (M,\omega)$ and L be semi-positive. Then the inequality* (6.3) *holds.*

Here the n dimensional Lagrangian submanifold L in (M,ω) is called semi-positive, if $\omega(\beta) \leq 0$ for any β with

$$3 - n \leq \mu_L(\beta) < 0.$$

Note that if $n \leq 3$, the semi-positivity automatically holds. This condition plays a role similar to the case of absolute case. The reason why we need to assume the semi-positivity is to handle the negative multiple cover problem. We recall that we should use $\mathbf{Z}/2\mathbf{Z}$-coefficient to have the cancellation of quantum effects in general which forces us to use integral cycles rather than rational cycles. We would like to emphasize that since we use $\mathbf{Z}/2\mathbf{Z}$-coefficients, we do not have to assume our Lagrangian submanifold is relatively spin.

The third application is so called the Maslov class conjecture. The general folklore conjecture says that the Maslov class $\mu_L \in H^1(L;\mathbf{Z})$ of Lagrangian embedding $L \subset \mathbf{C}^n$ is non-trivial for any compact Lagrangian embedding in $\mathbf{C}^n$. (We note that if the ambient symplectic manifold (M,ω) satisfies $c_1(TM) = 0$, then μ_L can be regarded as an element of $H^1(L,\mathbf{Z})$.) We can give a new partial answer.

Theorem 6.5. *Let L be a compact embedded Lagrangian submanifold of $\mathbf{C}^n$ that satisfies $H^2(L;\mathbf{Z}/2\mathbf{Z}) = 0$. Then its Maslov class $\mu_L \in H^1(L;\mathbf{Z})$ is nonzero.*

Moreover we can show the following estimate.

Theorem 6.6. *Let L be a compact embedded Lagrangian submanifold of $\mathbf{C}^n$. Suppose that it is unobstructed in the sense of Definition 5.2. Then we have the following inequality;*

$$1 \leq \Sigma_L \leq n+1.$$

Here Σ_L *is a non-negative integer defined by* Image $(\mu_L) = \Sigma_L \mathbf{Z}$, *where* μ_L *is the Maslov index homomorphism in* (2.0).

References.

[ASKZ] M. Alexandrov, A. Schwarz, M. Kontsevich and O. Zaboronsky, The geometry of the master equation and topological quantum field theory, *Intern. J. Modern Phys.* A, **12** (1997) 1405 - 1429.

[BV] I. Batalin and G. Vilkovsky, Quantization of gauge theories with linearly dependent generators, *Phys. Lett.* B, **311** (1993) 123 - 129.

[B] K. Behrend, Gromov-Witten invariants in algebraic geometry, *Invent. Math.*, **127** (1997) 604 - 617.

[FOOO] K. Fukaya, Y-G Oh, H. Ohta and K. Ono, Lagrangian intersection Floer theory – Anomaly and Obstruction –, to appear.

[FO] K. Fukaya and K. Ono, Arnold conjecture and Gromov-Witten invariants, *Topology* **38** (1999) 933 - 1048.

[GJ] E. Getzler and J. Jones, A_∞ algebra and cyclic bar complex, *Illinois J. Math.* **34** (1990) 256 - 283.

[K] M. Kontsevich, Deformation quantization of Poisson manifolds, q-alg/9709040

[LT] G. Liu and G. Tian, Floer homology and Arnold conjecture, *J. Diff. Geom.* **49** (1998) 1 - 74.

[Oh] Y.-G. Oh, Floer cohomology of Lagrangian intersections and pseudo-holomorphic disks I, II, *Comm. Pure and Appl. Math.* **46** (1993) 949 - 994 and 995 - 1012. *Addenda, ibid,* **48** (1995), 1299 - 1302.

[Sch] V. Schechtman, Remarks on formal deformations and Batalin-Vilkovsky algebras, math/9802006

[St] J. Stasheff, Homotopy Associativity of H-Spaces I, II, *Trans. Amer. Math. Soc.* (1966) 275 - 312.